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Random matrix ensembles of time-lagged correlation matrices: Derivation of eigenvalue spectra and analysis of financial time-series

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We derive the exact form of the eigenvalue spectra of correlation matrices derived from a set of time-shifted, finite Brownian random walks (time-series). These matrices can be seen as real, asymmetric random matrices where the time-shift superimposes some structure. We demonstrate that for large matrices the associated eigenvalue spectrum is circular symmetric in the complex plane. This fact allows us to exactly compute the eigenvalue density via an inverse Abel-transform of the density of the *symmetrized* problem. We demonstrate the validity of this approach numerically. Theoretical findings are next compared with eigenvalue densities obtained from actual high frequency (5 min) data of the S&P500 and discuss the observed deviations. We identify various non-trivial, non-random patterns and find asymmetric dependencies associated with eigenvalues departing strongly from the Gaussian prediction in the imaginary part. For the same time-series, with the market contribution removed, we observe strong clustering of stocks, into *causal* sectors. We finally comment on the stability of the observed patterns.

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1. INTRODUCTION

One of the pillars of contemporary theory of financial economics is the notion of correlation matrices of timeseries of financial instruments; the capital asset pricing model [1] and Markowitz portfolio theory [2] probably being the most prominent examples. Recent empirical analyses on the detailed structure of financial correlation matrices have shown that there exist remarkable deviations from predictions that would be expected from the efficient market hypothesis. In particular, based on pioneering work [3, 4], eigenvalue spectra of empirical equal-time covariance matrices have been analyzed and compared to predictions of eigenvalue densities for Gaussian-randomness obtained from random matrix theory (RMT). It has been shown, that the eigenvectors which strongly depart from the spectrum obtained by RMT contain information about sector organization of markets [5, 6]. The largest eigenvalue has been identified as the 'market-mode', and it has been pointed out that a 'cleaning' of the original correlation matrices by removing the noise part of the spectrum explainable by RMT results in an improved mean variance efficient frontier which seems to be much more adequate than the one obtained by Markowitz (see e.g. the recent discussion in [7]). Further, RMT provides an almost full understanding of why the Markovitz approach is close to useless (dominance of small eigenvalues which lie in the noise regime) in actual portfolio management.

Initially, RMT has been proposed to explain energy

spectra of complicated nuclei half a century ago. In its simplest form, a random matrix ensemble is an ensemble of $N \times N$ matrices \mathbf{M} whose entries are uncorrelated iid random variables, and whose distribution is given by

$$P(\mathbf{M}) \sim \exp \left(-\frac{\beta N}{2} \text{Tr}(\mathbf{M}\mathbf{M}^T) \right), \quad (1)$$

where β takes specific values for different ensembles of matrices (e.g. depending on whether or not the random variables are complex- or real-valued). Eigenvalue spectra and correlations of eigenvalues in the limit $N \rightarrow \infty$ have been worked out for *symmetric* $N \times N$ random matrices by Wigner [8]. For real valued matrix entries, such symmetric random matrices are sometimes referred to as the Gaussian orthogonal ensemble (GOE).

The symmetry constraint has later been relaxed and the probability distributions of different ensembles (real, complex, quaternion) – known as the Ginibre ensembles (GinOE, GinUE, GinSE) – have been derived [9] in the limit of infinite matrix size. For ensembles of random real asymmetric matrices (GinOE) – the most difficult case – progress has only slowly been made under great efforts over the past decades. The eigenvalue density could finally be derived via different methods [10, 11], where – quite remarkably – the finite-size dependence of the ensemble has also been elucidated [11, 12].

However, these developments in RMT do not yet take into account the timeseries character of financial applications, i.e. the fact, that one deals – in general – with (lagged) covariance matrices stemming from finite rectangular $N \times T$ data matrices \mathbf{X} , which contain data for N different assets (or instruments) at T observation points. The matrix ensemble corresponding to the $N \times N$ covariance matrix $\mathbf{C} \sim \mathbf{X}\mathbf{X}^T$ of such data is known as the Wishart ensemble [13] and is a cornerstone of multivari-

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ate data analysis. For the case of uncorrelated Gaussian distributed data, the exact solution to the eigenvalue-spectrum of \mathbf{XX}^T is known as Marcenko-Pastur law (for $N \rightarrow \infty$) and has been used as a starting point for random matrix analysis of correlation matrices at lag zero [3, 4, 5, 6, 7, 14]. Moreover a quite general methodology of extracting meaningful correlations between variables has been discussed based on a generalization of the Marcenko-Pastur distribution [15]. The underlying method was the powerful tool of singular-value decomposition and RMT was used to predict singular-value spectra of Gaussian randomness.

The time-lagged analogon to the covariance matrix is defined as $\mathbf{C}_\tau \sim \sum_t^T r_t^i r_{t-\tau}^j$, where one timeseries is shifted by τ timesteps with respect to the other. In contrast to (real-valued) equal-time correlation matrices of the Wishart ensemble, which have a real eigenvalue spectrum, the spectrum of \mathbf{C}_τ is defined in the complex plane since matrices of these type are in general asymmetric. While the complex spectrum of \mathbf{C}_τ remains unknown so far, results for *symmetrized* lagged correlation matrices have been reported recently [16, 17]. However, it has to be noted that treating the symmetrized object $\mathbf{C}_\tau^S = \frac{1}{2}(\mathbf{C}_\tau + \mathbf{C}_\tau^T)$ [16, 17] is problematic from a practical viewpoint, since one may lose important causal information via the symmetrization. Especially asymmetric dependencies (where e.g. the entry C_τ^{ij} carries non-random information and C_τ^{ji} is just noise) are prone to be lost when turning to the analysis of the object \mathbf{C}_τ^S since $\frac{1}{2}(C_\tau^{ij} + C_\tau^{ji})$ may be classified random although C_τ^{ij} carries information. The problem of working out actual predictions based on symmetrized lagged correlations (and not on the initial asymmetric ones) would be even more embarrassing from a practical viewpoint since $\frac{1}{2}(C_\tau^{ij} + C_\tau^{ji})$ will, in general, be a bad predictor for C_τ^{ij} .

Moreover, it is the analysis of the initial asymmetric time-lagged correlations which forms a fundamental part of finance and econometrics, and which has attracted considerable attention in the respective literature. The existence of asymmetric lead-lag relationships has been initially reported for the U.S. stock market [18]. Specifically, it was found that returns of large stocks lead those of smaller ones. Later, trading volume was identified as a significant determinant of such lead-lag patterns, and returns of high-volume stocks (portfolios) were found to lead those of low-volume stocks (portfolios) [19]. These lead-lag effects have primarily been explained by different effects of information adjustment asymmetry. For instance, a model was brought forward in [20], where it was argued, that, as soon as previous price changes are observed and marketwide information can thus be incorporated in the marketmakers' evaluation of stock prices, lagged correlations may emanate. Another type of information asymmetry can be seen in the different number of investment analysts following a firm's stock price [21]. Other explanatory approaches, include the institutional ownership of stocks [22], the different exposure of stocks to persistent factors [23], or transaction costs and market

microstructure [24] as causes of lagged autocorrelations. Whether or not non-synchronous trading may constitute a source of lead-lag relationships or not is an issue of ongoing discussion [18, 25, 26]. Recently, aiming at a closer empirical understanding of lagged correlations, the dependence of the strength of lagged correlations on the chosen time-shift τ has been analyzed for high-frequency NYSE data [27]. It was shown, that the lagged correlation function typically exhibits an asymmetric peak. The revealed patterns basically showed structures consistent with those found in [18] (e.g. patterns where more 'important' companies pull smaller, less 'important' ones). Interestingly, also evidence for a diminution of the Epps effect [28] has been demonstrated based on lagged cross-correlations of NYSE-data, as lead-lag dependencies seem to diminish over the years [29].

As diverse and interesting these approaches are, the methods applied mainly focus on Granger causality, vector autoregressive models and shrinkage estimators. In this paper, we extend the methodology to eigenvalue analysis of time-lagged correlations. First, we discuss how solutions of RMT problems pertaining to real, asymmetric matrices can be obtained from solutions to the symmetrized problem via an inverse Abel-transform. The respective developments will then enable us to derive the form of the eigenvalue spectra of the pure random case. As an immediate application we compare these theoretical results with real financial data and relate the observed deviations to market specific features.

The paper is organized as follows: In Section 2 we fix the notation and develop the spectral form of asymmetric real random correlation matrices. In Section 3 we apply the introduced methodology to empirical correlation matrices of 5 min log-returns of the S&P500 and discuss the meaning of deviant eigenvalues from several perspectives. Time-dependence issues are discussed in Section 4 and in Section 5 we finally conclude.

2. SPECTRA OF TIME-LAGGED CORRELATION MATRICES

2.1. Notation

The entries in the $N \times T$ data matrices \mathbf{X} for N assets and T observation times, are the log-return time-series of asset i at observation times t ,

$$r_t^i = \ln S_t^i - \ln S_{t-1}^i, \quad (2)$$

after subtraction of the mean and normalization to unit variance, i.e. division by $\sigma_i = \sqrt{\langle (r_t^i)^2 \rangle - \langle r_t^i \rangle^2}$. Here, S_t^i is the price of asset i at time t . One time unit is the time difference between observations at $t+1$ and t , e.g. a day, 5 minutes; for tic data it can also be of variable size. Time-lagged correlation functions of unit-variance log-return series among stocks are defined as

$$\mathbf{C}_\tau(T) \equiv \langle (r_t^i - \langle r_t^i \rangle)(r_{t-\tau}^j - \langle r_{t-\tau}^j \rangle) \rangle_T, \quad (3)$$

where the time-lag τ is measured in time units and $\langle \dots \rangle_T$ stands for a time-average over the period T . We drop (T) in the following, except for Section 4. Equal-time correlations are obviously obtained for $\tau = 0$. For $\tau \neq 0$, the lagged correlation matrix \mathbf{C}_τ is generally not symmetric and contains the lagged autocorrelations in the diagonal. It can be written as

$$\mathbf{C}_\tau = \frac{1}{T} \mathbf{X} \mathbf{D}_\tau \mathbf{X}^T, \quad (4)$$

where $\mathbf{D}_\tau \equiv \delta_{t,t+\tau}$ and where \mathbf{X} is the $N \times T$ normalized time-series data. Denoting the eigenvalues of \mathbf{C}_τ by λ_i and their associated right eigenvectors by \tilde{u}_i (or $\mathbf{U} = u_{ik}$), where $i, k = 1, \dots, N$, we may write the eigenvalue problem as $\mathbf{C}_\tau \tilde{u}_j = \lambda_j \tilde{u}_j$. We immediately recognize that eigenvalues λ_i are either real or complex conjugate, since the matrix elements of \mathbf{C}_τ are real and thus the conjugate eigenvalue λ_i^* also solves the eigenvalue problem. In the following we refrain from discussing the case of left eigenvectors since this would not yield additional insights. Regarding the elements of \mathbf{C}_τ as random variables with a certain distribution, we should keep in mind that their specific construction, Eq. (4), results in a departure from a 'purely' random real asymmetric $N \times N$ matrix where the entries are iid Gaussian distributed. Thus we do – in general – *not* expect a flat eigenvalue distribution as in the Ginibre-Girko case. Rather, we can interpret \mathbf{C}_τ as a random real asymmetric matrix with a special structure due to its construction. In general, comparably little work has been done to understand the eigenvalue spectra of such random real asymmetric matrices. However, it was shown that the problem can be treated in a way formally equivalent to classical electrostatics [30].

2.2. General Arguments

We start our original arguments by making use of the electrostatic analogy discussed in [30]. The idea is to interpret the distribution of eigenvalues in the complex plane as a distribution of electrical charges in 2 dimensions. The density of eigenvalues in the complex plane, $\rho(z) = \rho(x, y)$ with $z = x + iy$, can then be calculated by the Poisson equation

$$\rho(x, y) = -\frac{1}{4\pi} \Delta \phi(x, y). \quad (5)$$

In the present case, it can be verified easily that the corresponding potential in 2 dimensions is given by

$$\phi(x, y) = -\frac{1}{N} \langle \ln \det ((\mathbf{1}z^* - \mathbf{C}_\tau^T)(\mathbf{1}z - \mathbf{C}_\tau)) \rangle_c, \quad (6)$$

where $\langle \dots \rangle_c$ denotes the average over the distribution of \mathbf{X} and $\mathbf{1}$ denotes the identity matrix.

$$P(\mathbf{X}) \sim \exp \left(-\frac{N}{2} \text{Tr}(\mathbf{X} \mathbf{X}^T) \right). \quad (7)$$

We expand the argument of the determinant in Eq. (6) to obtain the positive definite matrix

$$\mathbf{H} = \mathbf{1}|z| + \mathbf{C}_\tau \mathbf{C}_\tau^T - x(\mathbf{C}_\tau + \mathbf{C}_\tau^T) + iy(\mathbf{C}_\tau - \mathbf{C}_\tau^T). \quad (8)$$

This form now shows that any symmetric (anti-symmetric) contribution of \mathbf{C}_τ^{ij} only influences the real (imaginary) part of z .

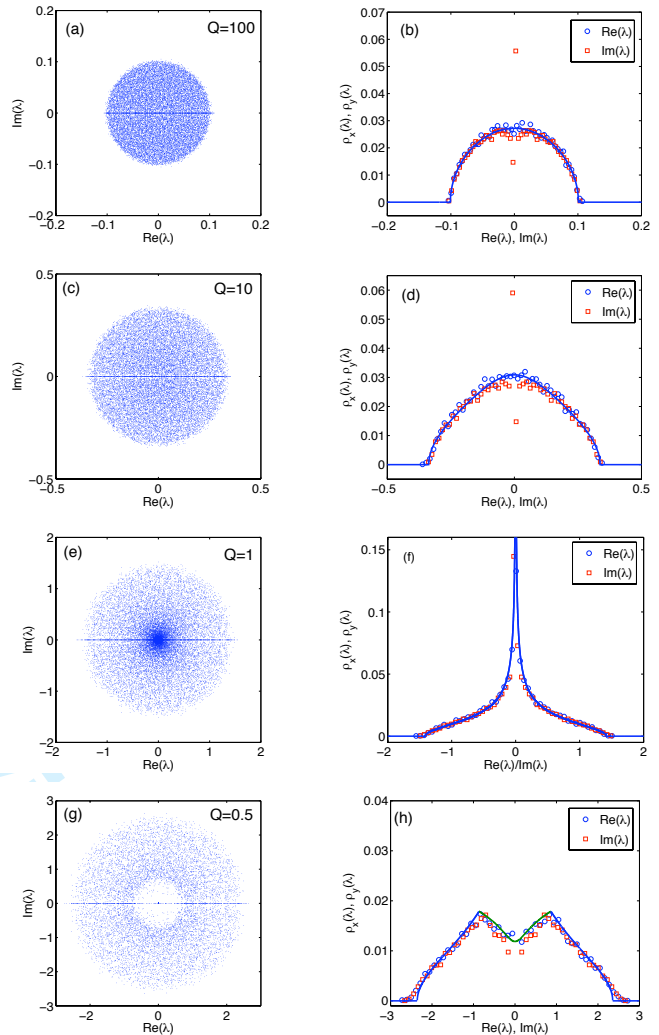


FIG. 1: Complex eigenvalue spectra of time-lagged correlation matrices, obtained from random matrices \mathbf{X} . The entries of \mathbf{X} are iid and Gaussian with unit variance. In (a), (c), (e) and (g) the position of the eigenvalues is shown in the complex plane for values of $Q \equiv \frac{T}{N} = 100, 10, 1$ and 0.5 , respectively. The visibly enhanced density along the real axis is the finite-size effect mentioned in the text. The right column shows the projections of the EVs onto the real and imaginary axis. The solid lines are the theoretically expected curves, which are numerical solutions to Eq. (17). Note in (h) that for this projection, the eigenvalue spectra is composed of different solutions to Eq. (17) as $G(z)$ itself has a discontinuity. The divergence at $z = 0$ is not shown for analytical curves associated with $Q = 100, 10$ and 0.5 .

Because of the absence of any structural difference in the randomness of the symmetric and the anti-symmetric part of matrix \mathbf{C}_τ , the expression of Eq. (8) is equivalent under exchange of x and y in the distributional sense and Eq. (5) will thus be a symmetric function in x and y . Since one does not expect any direction in the complex plane being distinguished from any other in the limit $N \rightarrow \infty$, this leads to the central argument of our discussion, namely that the potential is a function of the radius $r = \sqrt{x^2 + y^2}$, $\phi(x, y) = \phi(r)$ in the limit $N \rightarrow \infty$. Thus also the eigenvalue density resulting from (6) will be a radial symmetric function,

$$\rho(x, y) = \rho(r) \equiv \frac{1}{2\pi r} \int_{\mathcal{S}} dz \rho(z) \delta(|z| - r) \quad (9)$$

A more formal argument can be given via expanding the matrix \mathbf{H} entering the potential. Since the entries in \mathbf{C} are typically smaller than one, \mathbf{H} can be written as $\mathbf{H} \approx |z|(\mathbf{1} + \epsilon \mathbf{B})$. Here, ϵ is a small perturbation, and $\mathbf{B} = \mathbf{C}\mathbf{C}^T/|z| - \bar{x}(\mathbf{C} + \mathbf{C}^T) + i\bar{y}(\mathbf{C} - \mathbf{C}^T)$ with $\bar{x} = x/|z|$ and $\bar{y} = y/|z|$. We fix $|z| = 1$ without loss of generality and write the determinant as a Taylor series,

$$\begin{aligned} \phi(x, y) &= -\frac{1}{N} \langle \ln \det(\mathbf{H}) \rangle_c = -\frac{1}{N} \langle \text{Tr} \ln(\mathbf{H}) \rangle_c \\ &\approx -\frac{1}{N} \langle \text{Tr}(\mathbf{B}) - \text{Tr}\left(\frac{\mathbf{B}^2}{2}\right) + \text{Tr}\left(\frac{\mathbf{B}^3}{3}\right) - \dots \rangle_c \end{aligned} \quad (10)$$

Based on this series, we checked up to fourth order that this expansion indeed only leads to terms in r for $N \rightarrow \infty$; we outline some aspects of the calculation in Appendix A.

Before turning to the main argument of our derivation, we discuss the support of the eigenvalue density. Clearly, if $\rho(r)$ is circular symmetric, the support \mathcal{S} of the eigenvalue-spectrum will be bounded by a circle and is thus definable via a maximal radius r_{max} . Since r_{max} is governed by the standard deviation of the underlying random matrix elements, one can compute the extent of the support of \mathbf{C}_τ by considering the support of symmetric (r_{max}^S) and anti-symmetric matrices (r_{max}^A). Let these be defined by $\mathbf{C}_\tau^S \equiv \frac{1}{2}(\mathbf{C}_\tau + \mathbf{C}_\tau^T)$ and $\mathbf{C}_\tau^A \equiv \frac{1}{2}(\mathbf{C}_\tau - \mathbf{C}_\tau^T)$. If we assume that the standard deviations of the symmetric and anti-symmetric matrices are equal, $\sigma_S = \sigma_A$, this implies that the standard deviation σ of the matrix \mathbf{C}_τ , will be $\sigma = \sqrt{2}\sigma_S/2$. Taking into account that one has two degrees of freedom in the two-dimensional case, the support of \mathbf{C}_τ can be defined via a disc with radius

$$r_{max}^S = \frac{1}{\sqrt{2}} r_{max}^S = \frac{1}{\sqrt{2}} r_{max}^A \quad (11)$$

We will now discuss a new method to determine $\rho(r)$ based on its radial symmetry. To our best knowledge, this method has not been discussed in literature so far and is applicable whenever the potential in Eq. (6) is radial symmetric. The method is based on the sensible

conjecture that the projections of $\rho(r)$ onto the x -axis, denoted by $\rho_x(\lambda)$, and the projection onto the y -axis, $\rho_y(\lambda)$, are nothing but the rescaled spectra of the solution to the symmetric, $\rho^S(\lambda)$, and to the anti-symmetric problem, $\rho^A(y)$. To be more explicit,

$$\begin{aligned} \rho_x(\lambda) &\equiv \rho(\text{Re}(\lambda)) = \int_{\mathcal{S}} \rho(r) dy = \rho^S(\sqrt{2}x) \\ \rho_y(\lambda) &\equiv \rho(\text{Im}(\lambda)) = \int_{\mathcal{S}} \rho(r) dx = \rho^A(\sqrt{2}y) \end{aligned} \quad (12)$$

where the integration extends over the support \mathcal{S} in the complex plane. Although this conjecture might seem quite natural we shall provide numerical evidence for its correctness below. Noting that the eigenvalue density of the symmetric problem can be obtained from the well-known relation

$$\rho^S(x) = \sum_n \delta(x - x_n) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} [\text{Im}(G^S(x - i\epsilon))] \quad (13)$$

the main idea of this work is now to note that one can use the inverse Abel-Transform to actually determine the radial symmetric density $\rho(r)$.

Indeed, since the rescaled eigenvalue density of the symmetrized problem $\rho^S(\sqrt{2}x)$ is nothing but the projection of $\rho(r)$ onto the real axis, Eq. (12), it can be written as the Abel-transform [34],

$$\rho^S(\sqrt{2}x) = 2 \int_x^\infty \frac{\rho(r)r}{\sqrt{r^2 - x^2}} dr \quad (14)$$

of the radial density $\rho(r)$. One can then reconstruct the desired eigenvalue spectrum *exactly* (in the limit $N \rightarrow \infty$) via the inverse Abel-transform, and thus via the cuts of the Greens function of the symmetric problem,

$$\rho(r) = -\frac{1}{\pi^2} \int_r^\infty \frac{\frac{d}{dx} \lim_{\epsilon \rightarrow 0} [\text{Im}(G_\tau^S(\sqrt{2}x - i\epsilon))]}{\sqrt{x^2 - r^2}} dx \quad (15)$$

Here, we have made use of Eq. (13). Since Eq. (15) can be problematic if evaluated numerically, we also specify a form which exploits the Fourier-Hankel-Abel cycle [34]

$$\rho(r) = 2\pi \int_0^\infty q J_0(2\pi r q) \int_{-\infty}^\infty \rho^S(x) e^{-2\pi i x q} dx dq \quad (16)$$

where $J_0(x)$ denotes the zeroth-order Besselfunction. We also note, that yet another method of determining $\rho(r)$ is the evaluation of the inverse Radon-transform of $\rho^S(\sqrt{2}x)$.

Eq. (15) represents a hitherto unknown way of attacking the problem of the determination of complex eigenvalue densities in the case of radial symmetry and for any radial symmetric eigenvalue density in the limit $N \rightarrow \infty$. Its extreme simplicity is demonstrated in Appendix B, where – as a specific and prominent example – we show the case of deriving the density of real asymmetric random matrices directly from the semi-circle law. We note

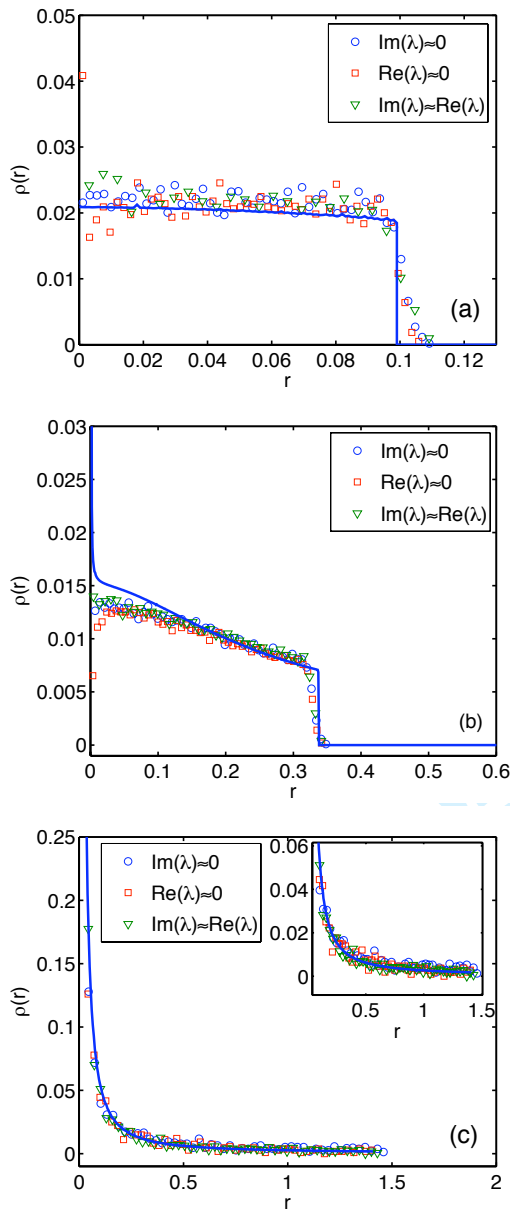


FIG. 2: Radial eigenvalue densities approximated via simulations along different directions (real axis, imaginary axis and the diagonal in the complex plane). Numerical data for finite matrices is compared with the solution of the inverse Abel-transform. (a) $Q=100$ (b) $Q=10$ (c) $Q=1$; the inset shows a detail of the curve.

that the solution of the symmetric problem will generally be valid only in the $N \rightarrow \infty$ limit. Thus, although the Abel-inversion gives an exact result, discrepancies may occur because of finite-size effects.

2.3. Application to lagged correlation matrices

We now turn to our specific problem of determining the eigenvalue density of \mathbf{C}_τ . What is left is to confirm the validity of our conjecture, Eq. (12), and to show, that – as a consequence – Eq. (15) gives an approximation to the radial eigenvalue distribution, $\rho(r)$. To start, we leave our original discussion for a short moment and refer to existing literature on the symmetric problem which we will use in our formalism: It has been shown that the Greens function $G(z)$ of the symmetrized problem $\mathbf{C}_\tau^S = \frac{1}{2T} \mathbf{X}(\mathbf{D}_\tau + \mathbf{D}_{-\tau})\mathbf{X}^T$ is given by the polynomial equation [16, 17],

$$\begin{aligned} \frac{1}{Q^3} z^2 G^4(z) - 2 \frac{1}{Q^2} \left(\frac{1}{Q} - 1 \right) z G^3(z) - \\ \frac{1}{Q} \left(z^2 - \left(\frac{1}{Q} - 1 \right)^2 \right) G^2(z) + 2 \left(\frac{1}{Q} - 1 \right) z G(z) + 2 - \frac{1}{Q} = 0 \end{aligned} \quad (17)$$

with $Q \equiv T/N$ playing the role of a information-to-noise ratio. We note that eigenvalue-densities for different values of Q have not been discussed so far and that we have verified that the Greens function pertaining to the asymmetric problem leads to eigenvalue densities on the complex axis functionally identical to the ones resulting from Eq. (17).

To verify our conjecture, we can calculate our $\rho_x(\lambda)$ from Eq. (17) by using Eqs. (13) and (12). Fig. 1 shows (simulated) spectra of $\mathbf{C}_{\tau=1}$ as defined by Eq. (4) with iid entries in the columns of \mathbf{X} , for various values of Q . Note, that for $Q < 1$ the shape of the boundary of eigenvalues in the complex plane changes from a disk to an annulus [41]. We immediately recognize that eigenvalues are enhanced along the real axis and that, as a consequence, the density is lower in the vicinity of the real axis. This can be attributed to a well-known finite-size effect [12, 30] which we confirmed for different values of Q (not shown). Of course, the finite-size effect implies that circular symmetry is not fully fulfilled for finite matrices of the GinOE [42]. Thus, we also expect to observe some discrepancies between the theoretical results based on the Abel-transform and the empirical densities of finite, lagged correlation matrices based on random data. In our concrete case, the prediction of the projections ρ_x and ρ_y (blue lines, obtained from Eq. (13) and Eq. (17)) depicted in the right column of Fig. 1 is in good agreement with the numerical data for the real parts of the eigenvalues (ρ_x). For the projection of the complex parts (ρ_y) we recognize that there is a slight deviation from the prediction (due to the enhanced density along the real axis). We also checked projections with data obtained via rotating all the individual eigenvalues in the complex plane for different angles. Apart from some minor effects attributable to the inhomogeneity around the real axis we found no significant discrepancies.

Thus, we have numerically verified our conjecture and may turn towards the point of reconstructing the radial eigenvalue density. The function to be transformed ($\rho^S(\sqrt{2}x)$ or $\rho^A(\sqrt{2}y)$) may be evaluated exactly (with

some effort) for the symmetric case from Eq. (13) and Eq. (17). The remaining integral Eq. (15) will, however, be hard to solve in general. Nonetheless, we were able to solve the case $Q = 1$ analytically and obtain the exact formula for the eigenvalue density,

$$\rho_{Q=1}(r) = \frac{1}{K} \left[2^{3/4} 3r \Gamma\left(\frac{5}{4}\right) \Gamma\left(\frac{5}{4}\right) \Phi_2\left(\frac{1}{4}, \frac{5}{4}, \frac{3}{2}, \frac{\lambda^2}{2}\right) - 2^{1/4} \Gamma\left(-\frac{1}{4}\right) \Gamma\left(\frac{7}{4}\right) \Phi_2\left(-\frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{\lambda^2}{2}\right) \right], \quad (18)$$

with $K \equiv 6\sqrt{\pi^5 r^3}$. Here, $\Gamma(x)$ denotes the Gamma function and $\Phi_2^1(a, b, c, z)$ the hypergeometric function; the derivation is briefly summarized in Appendix C. Note that $\lim_{Q \rightarrow 0} G_Q^S(z) = \frac{1}{z}$, whereas for $Q \rightarrow \infty$ we expect the Greens function and the eigenvalue density to converge to those of a random real asymmetric matrix without specific structure, i.e. a flat eigenvalue-density in the sense of [30].

We were not able to derive closed expressions for other values of Q , since already the solution of Eq. (17) results in lengthy expressions. In these cases we computed the integral Eq. (15) numerically. The results are depicted in Fig. 2 for $Q = 100$, $Q = 10$ and $Q = 1$. The theoretical predictions are accompanied by data obtained from performing cuts along various directions of the spectra $\rho(x, y)$ from Fig. 1, namely along the x-axis, the y-axis and along the diagonal direction, i.e. $\text{Re}(\lambda) = \text{Im}(\lambda)$. We performed these cuts numerically via calculating the density in narrow strips along the different directions. The theoretical prediction catches the different experimental densities very well. Especially for $Q = 100$ and $Q = 1$ results are consistent with the predictions to a high degree. For $Q = 10$ we observe some discrepancies for values $r < 0.1$. These are associated with the finite-size effect of enhanced eigenvalue density along the real axis discussed above. In fact, a closer investigation of the N -dependence of the fraction of real eigenvalues f_{real} reveals a scaling $f_{\text{real}} \sim N^{-1/2}$ quite independent of the value of Q (not shown). This scaling is equivalent to the GinOE case [11, 12, 30].

To shortly summarize our theoretical investigation: We have used a well-known analogy to classical electrostatics to present our original arguments and calculations concerning radial symmetry of the potential in the case of lagged correlation matrices. We have then introduced a novel method of calculating the radial eigenvalue-density in such cases via an inverse Abel-Transform and we used existing results for symmetrized lagged correlation matrices as input to our method to arrive at an understanding of the initial asymmetric case.

3. EMPIRICAL ANALYSIS

3.1. Data

With a theoretical concept of and some specific knowledge about the eigenvalue-spectra of time-lagged correlation matrices, we now turn to actual financial data and study empirical lagged correlation matrices \mathbf{C}_τ . We analyze 5 min data of the S&P500 in the time period of Jan 2 2002 – Apr 20 2004. After rigorous cleaning the data set \mathbf{X} consisted of $N = 400$ time-series at $T = 44720$ observation times each. Of course, the empirical time-series and its distribution-functions showed the usual properties of high-frequency stock-returns (not shown).

From \mathbf{X} we construct two surrogate data sets, one by removing the market mode, the other by a scrambling of data. As $\tau = 1$ remains unchanged during the rest of the paper, we will occasionally drop the subscript, $\mathbf{C}_1 = \mathbf{C}$.

Market mode removed data: It is well known that the spectrum of equal-time correlations is dominated by a single very large eigenvalue which can be attributed to the so-called 'market-mode' describing movements common to all stocks, see e.g. [5, 7, 36]. We define the market return (the movement of the index) by $r_t^m = \sum_{j=1}^N v_{1j} r_t^j$, where \tilde{v}_1 is the eigenvector associated with the largest eigenvalue $\tilde{\lambda}_1$ of the empirical covariance matrix at *equal times*, i.e. $\tau = 0$. To remove this market mode from the data we simply regress in the spirit of the Capital asset pricing model [1]

$$\tilde{r}_t = \tilde{\alpha} + \tilde{\beta} r_t^m + \tilde{\epsilon}_t, \quad (19)$$

where the residuals $\tilde{\epsilon}_t^i$ carry what is left of the structural information in the data; we denote this data set by \mathbf{X}^{res} , its elements being $X_{it}^{\text{res}} = \tilde{\epsilon}_t^i$.

Randomly reshuffled data: A reshuffled version \mathbf{X}^{scr} is generated by a random permutation of all elements of \mathbf{X} . This destroys all correlation structure but has exactly the same distributions as the original data. Correlation matrices from \mathbf{X}^{scr} should – up to potential non-Gaussian effects in the distributions – correspond to the developments in Section 2. We checked that the support of the eigenvalue-spectra pertaining to the lagged correlation matrices – which will be the quantity used for identifying deviating eigenvalues – indeed resembles the value r_{max} of the Gaussian case discussed in Section 2. We also checked the distribution of eigenvalues within the support and did not find strong deviations from the theoretical predictions for the empirical (non-Gaussian) return-series. This assures us, that we can still use the gaussian results to test whether or not the eigenvalues in the support can be regarded as noise. A treatment of the exact spectra of lagged correlation matrices of random Levy distributed data is, of course, far beyond the scope of the present work.

The following empirical investigation is based on these three datasets. We will first investigate whether or not

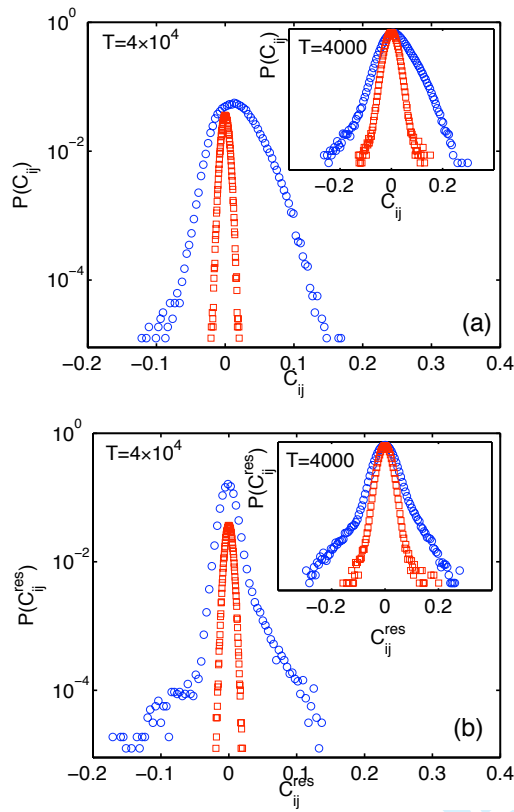


FIG. 3: (a) Empirical distribution $P(C_{ij})$ of the lagged correlation matrix elements C_{ij} for a sampling period of $T = 40000$ and $T = 4000$ (inset). Circles represent empirical data, red squares the situation for randomly reshuffled data from \mathbf{X}^{scr} . (b) shows the same for the removed market case, i.e. from \mathbf{X}^{res} . Individual frequencies are normalized by the summed frequencies for each plot.

\mathbf{C}_1 contains non-random information. Comparing the empirical eigenvalue spectra to the theoretical ones we will then discuss the source of differences between the two and we will subsequently show in detail how departing eigenvalues may be interpreted.

3.2. Empirical time-lagged financial random matrices

As a first investigation of the data, we show the distribution of matrix elements $P(C_1^{ij})$ of the empirical correlation matrix \mathbf{C}_1 (circles) in Fig. 3. Fig. 3a is based on \mathbf{X} and Fig. 3b on \mathbf{X}^{res} . Squares show the results for the randomly reshuffled data $\mathbf{C}_1^{\text{scr}}$. The inset shows the result for a shorter sampling time of $T = 4000$. Clearly, there is 'significant' correlation in the data in both cases, contrasting the Gaussian prediction of the efficient market hypothesis. We mention that the observed correlations typically decrease with decreasing observation frequency (e.g. examining hourly or daily returns) and also decrease with increasing timeshift τ [43].

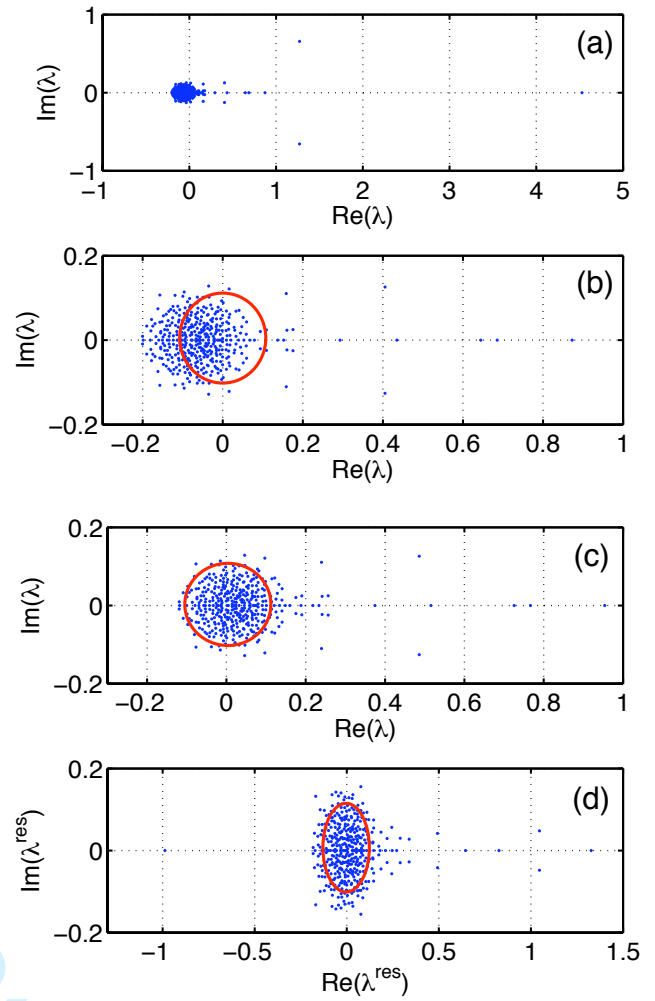


FIG. 4: Eigenvalue spectra of lagged correlation matrices from 5 min S&P500 data. (a) shows the full spectrum with one very large deviation on the real axis ($\lambda_1 \sim 4.6$), and a large departing eigenvalue pair $\lambda_2 = \lambda_3^*$. (b) is a detail, clearly showing that the spectrum is shifted with respect to the 'bulk-disc'. (c) spectrum corrected for displacement d as discussed in the text. (d) is the eigenvalue spectrum based on the market removed data, \mathbf{X}^{res} , also after displacement correction. The circles in plots (b)–(d) indicate the theoretical support discussed in Section 2.

The situation for the market removed data \mathbf{X}^{res} , Fig. 3b, shows that lagged correlations are not distributed randomly as well. The frequency of higher values of \mathbf{C}_1 is slightly reduced and the curve has significantly changed shape. In the semilogarithmic plot of Fig. 3, the positive regime is clearly not following a square-polynomial curvature, but rather an exponential one. This also applies to the data sampled from $T = 4000$ subperiods, depicted in the inset of Fig. 3. Both empirical distribution functions also exhibit clear non-random negative *autocorrelations* which are the predominant source of the non-Gaussian tails for negative entries.

3.2.1. Eigenvalue spectra

We will now investigate if and how the non-random lagged correlations manifest themselves in the eigenvalue spectra of \mathbf{C}_1 . Fig. 4a-c shows the eigenvalue spectrum obtained from \mathbf{C}_1 at various stages. In Fig. 4a a few very strong deviations from the bulk of the eigenvalues are seen, most significantly one real eigenvalue $\lambda_1 \approx 4.6$ and a conjugate pair of complex eigenvalues. Fig. 4 (b) is a detail of (a) where a clear shift of the bulk of the eigenvalues with respect to the Gaussian regime (circle) is observed.

Correction of the shift: Closer analysis shows that this shift can be attributed to two effects: First, each deviating positive real eigenvalue $\tilde{\lambda}_i$ is associated with a shift s of the 'bulk' spectrum of $s \approx -\text{Re}(\tilde{\lambda}_i)/N$ in direction of the negative real axis. ('Departing' eigenvalues are those which have real parts larger than the radius of the theoretical support.) The shift of the 'disc' pertaining to this effect is then the sum of all effects from departing eigenvalues, $s_{\text{tot}} = -\frac{1}{N} \sum \tilde{\lambda}_i \text{Re}(\tilde{\lambda}_i) \approx -0.031$. A second contribution of the shift is due to the non-zero diagonal entries of the correlation matrices \mathbf{C}_1 . The shift of the center of the disk explainable by the mean of the diagonal elements is $\bar{C}_{11}^{ii} = -0.029$, such that the overall displacement is $d = s_{\text{tot}} + \bar{C}_{11}^{ii} = -0.060$. When corrected for the total shift we arrive at Fig. 4 (c). We repeated the same procedure for $\mathbf{C}_1^{\text{res}}$, obtaining $d^{\text{res}} = 0.081$. The resulting – displacement corrected – distribution is directly depicted in Fig. 4 (d). In both cases the shift of the center of the support is explained by our analysis and thus does not associated to non-random features of interest here (i.e. non-randomness in the lagged correlations).

After having explained the observed shift in the eigenvalue distribution, the next step is to analyze the shape of the distribution within the theoretical support. This will allow to decide whether or not the 'bulk' of the spectrum is random. To do so, we compare the predictions from Section 2 with the empirical data in Fig. 5. We show the projections of the empirical eigenvalues on the real and imaginary axis. The inset shows the theoretical prediction of the radial density integrated over the complex plane, $2r\pi\rho(r)$, compared with the empirical data, $\rho(|\lambda|)$. We chose this 'integrated' representation since data quality would be unsatisfying otherwise. The empirical spectra are truncated at $\text{Re}(\lambda) = 1$. Given the modest eigenvalue statistics ($N_\lambda = 400$) and the strong deviations outside the theoretical support, the agreement between the theoretical predictions for Gaussian noise and the empirical data seems rather satisfying and one can conclude that the factors represented in the bulk of the observed spectrum are indeed predominantly random. The eigenvalues lying outside the random regime can be confidently associated with specific non-random structures which will now be subject to closer examination.

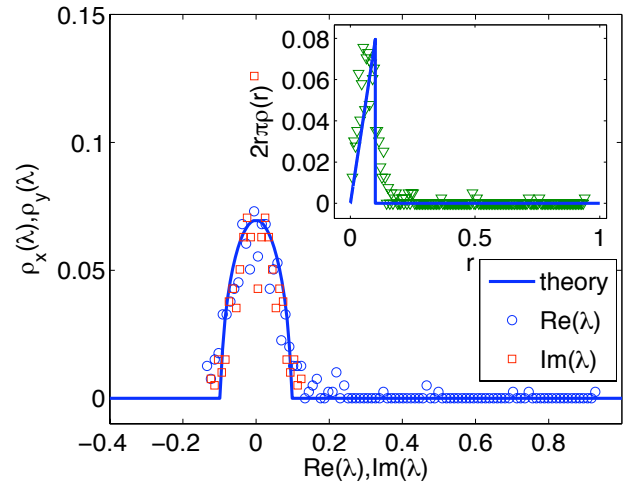


FIG. 5: Projection of the empirical spectrum pertaining to Fig. 4c on the real and imaginary axis. The blue line is the analytical solution discussed in Section 2. The inset shows the empirical distribution of $\rho(|\lambda|)$ compared with the analytical analogue $2r\pi\rho(r)$.

3.2.2. Interpretation of deviating eigenvalues

Strong deviations from the theoretical pure random prediction indicate significant correlation structure in the data. The nature of these deviations can be interpreted considering the form of the potential and its argument, Eq. (6) and Eq. (8). In these equations, symmetric non-randomness would affect the real part and asymmetric non-randomness the imaginary part of the potential and of the eigenvalue distribution). Eigenvalues departing on the real axis with no or only a small imaginary part will therefore be the effect of symmetric correlations. Complex conjugate eigenvalues departing on the imaginary axis will be attributable to asymmetric correlations. We stress again that non-random factors associated to such complex eigenvalues may be lost when analyzing the symmetrized matrix.

Thus, the departures of the largest eigenvalue in Fig. 4a and Fig. 4c are caused by a symmetric lagged correlation structure since this eigenvalue is real. Turning to the second and third largest eigenvalue, we see significant non-symmetric correlations in \mathbf{X} reflected in complex-conjugate pairs of eigenvalues with relatively large imaginary parts. The residuals \mathbf{X}^{res} exhibit a large negative real eigenvalue indicating symmetric *anti*-correlations between stocks. Such a departure is not visible for \mathbf{X} .

For a closer inspection of which assets 'participate' in a given eigenvector associated to a deviating eigenvalue, we define the inverse participation ratio for the eigenvectors \vec{u}_i in the form

$$\text{IPR}(\vec{u}_i) \equiv \sum_{k=1}^N |u_{ik}|^4 \quad . \quad (20)$$

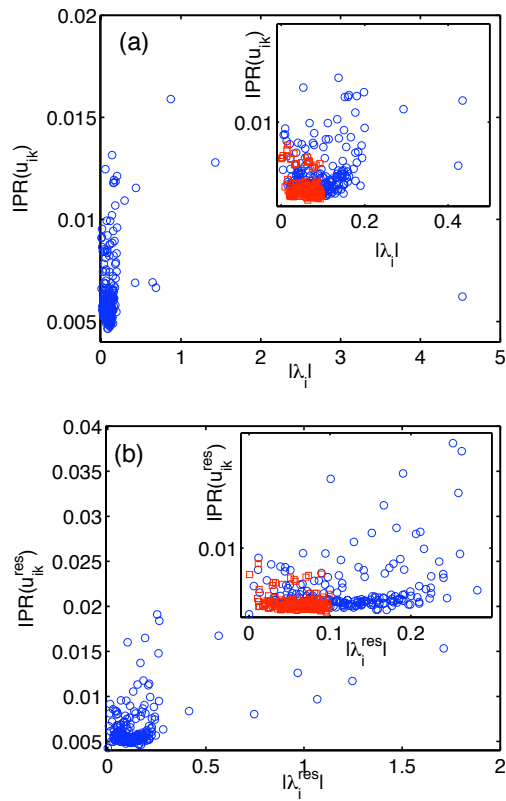


FIG. 6: (a) Inverse participation ratio as defined in Eq. (20) as a function of the absolute value of λ_i . Circles represent data from the empirical matrix, squares (inset) data from a random analogue, obtained from iid gaussian distributed \mathbf{X} . (b) The same as above but for eigenvectors obtained from the data with the market mode subtracted out.

This ratio shows to which extent each of the $N = 400$ assets contribute to the eigenvector \vec{u}_i . While a low IPR means that assets contribute equally, a large IPR signals that only a few assets dominate the eigenvector.

Fig. 6a shows the IPRs for the empirical correlation matrix \mathbf{C}_1 . The inset is a detail and also exhibits the IPRs of the randomly reshuffled data (squares). Clearly, the 'random' regime is not confined to an approximately constant region of IPRs but varies quite widely. This is in contrast to the symmetric case where one has a constant IPR for eigenvalues stemming from Gaussian randomness. We checked that the fluctuations observed here are already present in the Ginibre ensemble and are thus not associated to the specific structure of \mathbf{C}_τ . It is clear, that the IPRs belonging to the random case not being bound to a line hinders the identification of the eigenvectors with strong influence from only a few components to certain extent. However, one can nonetheless observe that the largest departing eigenvalue λ_1 is characterized by a rather small IPR, indicating an influence of a large number of assets. In contrast, some other deviant eigenvalues lie well above the random regime indicating the

Sector	GICS No. of Stocks N_s	
Energy	10	22
Materials	15	27
Industrials	20	44
Consumer Discretionary	25	63
Consumer Staples	30	35
Healthcare	35	40
Financials	40	71
Information Technology	45	63
Telecommunication	50	11
Utilities	55	24

TABLE I: Global Industry Classification Standard (GICS code), for the 10 main sectors of the S&P500 with the number of stocks in these sectors, see www.standardandpoors.com.

influence of only few stocks.

Again, we compare this situation with the one found for the residuals \mathbf{X}^{res} which is given in Fig. 6b. On average, the IPRs of the deviating eigenvalues are larger than in Fig. 6a, indicating increased presence of clustered structures. Having evidence of group structure in the lagged-correlations at hand, we now take a closer look at these structures.

3.2.3. Sector organization in time-lagged data

It is well known from RMT applications to covariance matrices ($\tau = 0$) of financial data, that the eigenvectors \vec{u}_i of large eigenvalues can be associated with the sector organization of markets. Let us label the different sectors with s , and define

$$\Delta_{sk} = \begin{cases} 1 & \text{if stock } k \text{ belongs to sector } s \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

To visualize the influence of each sector s to a given eigenvector i , we calculate

$$I_{si} \equiv \frac{1}{N_s} \sum_{k=1}^N \Delta_{sk} |u_{ik}|^2, \quad (22)$$

where N_s is the number of stocks in the respective sector, s . We evaluate Eq. (22) for the S&P500, using the standard sector classification scheme, the so-called GICS code, which is summarized in Table I. Focusing on some selected eigenvalues, Fig. 7 shows the contributions of the sectors in the case of the original (left column) and the market-mode removed data (right column). In the case of the original data, the information technology sector plays a decisive role for the largest 3 eigenvalues, namely λ_1 and $\lambda_2 = \lambda_3^*$. One can conclude that this sector thus explains a large part of the most distinctive non-random (symmetric and asymmetric) structure in \mathbf{C}_1 . For other eigenvectors, as for example λ_4 and λ_{10} as well as others

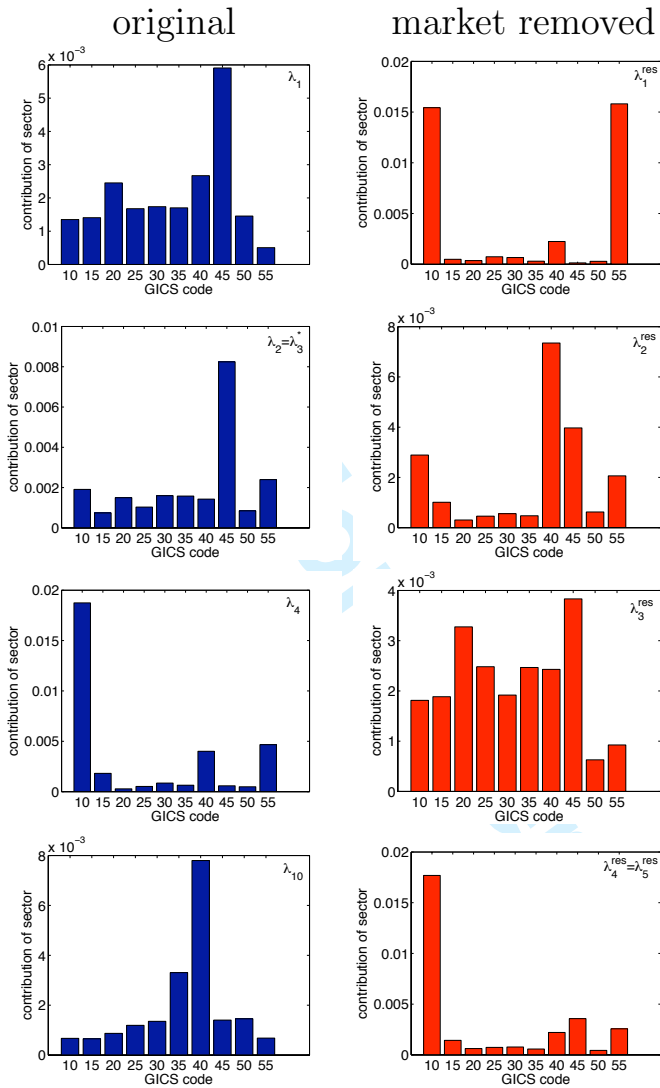


FIG. 7: Strength of participation, I_{si} , of the ten main sectors of the S&P500 (according to the GICS code) to eigenvectors \tilde{u}_i for some selected eigenvalues λ_i .

not shown here, a distinctive role is played by the energy and financial sector.

Turning towards the data where the market mode has been removed, we see that the largest eigenvalue λ_1^{res} is associated with a strong participation of the energy and utility sectors. In the second eigenvector, the financial sector is dominant, whereas the eigenvalue associated with the strong negative departure on the real axis, $\lambda_3 \approx -1$, is not dominantly influenced by any sector. For $\lambda_4 = \lambda_5^*$ we find a strong influence of the energy sector. Other eigenvectors also indicate a strong sectorial contribution (not shown).

We will now proceed to a network view to visualize and further discuss the findings of strong sectorial contribution associated to the deviating eigenvalues.

3.2.4. Lead-lag networks

Comparing eigenvalue spectra of the residuals with those of the initial data (Fig. 4), it is apparent that the market mode has a clear influence on the deviations and that the largest eigenvalue for the residuals is significantly reduced. One would expect that removing the (equal-time) market-mode also eliminates much of the correlations pertaining to small firms driven by large companies or similar 'star-like' structures (i.e. any network structure where one stock leads or lags many other stocks). In Fig. 8a we show a network view of the \mathbf{C}_1 correlation matrix, where a link is drawn for any $C_1^{ij} > 0.09$; Fig. 8b is the same after removal of the market mode, and removing entries $C_1^{ij \text{ res}} > 0.033$. The chosen cut-offs can be motivated from Fig. 3 by extracting the values of the C_1^{ij} where the distributions change their functional behavior.

Clearly, while in Fig. 8a there is not much clustering (except maybe for the utility sector), the market mode removed scenario in Fig. 8b exhibits distinctive clustering. As in the previous section, we mapped the nodes to the 10 most important sectors in the market. Nodes are colored according to these sectors in Fig. 8 along the lines of the accompanying color scheme. Clearly, the identified clusters correspond very nicely with industry sectors, as was found quite some time ago for the case $\tau = 0$.

The structures in the network-view can also be associated with the departing eigenvalues via decomposition of the lagged correlation matrix in its right eigenvectors,

$$\mathbf{C}_{\lambda_i} = \mathbf{U} \mathbf{\Lambda}_i \mathbf{U}^{-1}, \quad (23)$$

where $\mathbf{\Lambda}_i = \text{diag}(\lambda_i)$ denotes a diagonal matrix with only one entry at the respective position, associated with eigenvalue λ_i .

In the case of the original data and the contribution \mathbf{C}_{λ_1} of the largest eigenvalue to the lagged correlation matrix, we find assets from the Information Technology (IT) sector leading stocks of different sectors (not shown) with positive lagged correlations in the $\mathbf{C}_{\lambda_1}^{ij}$. Quite similarly, the most prominent features of the conjugate pair $\lambda_2 = \lambda_3^*$ are found to be associated with a hub-like influence of the IT sector when examining $\mathbf{C}_{\lambda_2}^{ij}$. Networks pertaining to λ_4 and λ_{10} primarily exhibit intersectorial ties of the Energy and Financial sector, where we also observed hub-like anti-correlations pointing from stocks of the Financial sector to the Energy sector. For all eigenvalues, we found no indication for the leading stocks being the ones with the highest market capitalization as would be implied by the finding of [18].

Turning to an examination of the residuals \mathbf{X}^{res} and the correspondence between the $\mathbf{C}_{\lambda_i}^{\text{res}}$ and the structure observed in the network-representation we found very nice correspondence: The contribution of the largest eigenvalue λ_1^{res} to the matrix \mathbf{C}^{res} shows a strong clustering of the Energy & Utility sector, which is visible in

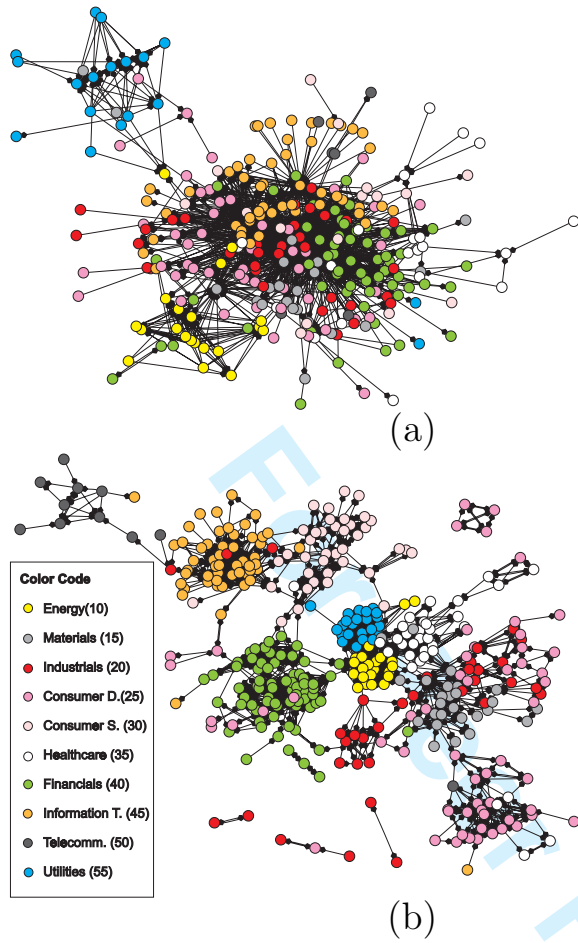


FIG. 8: Network view of \mathbf{C}_1 (a). A Link was drawn for $C_1^{ij} > 0.09$. The situation for the regressed scenario is shown in (b) with a threshold of $C_1^{ij \text{ res}} > 0.033$. All network pictures resulted from a Kamada-Kawai algorithm.

Fig. 8b as well. The fact that practically no assets apart from the Energy and Utilities sector are represented is also fully conforming with the top right panel of Fig. 7. $\mathbf{C}_{\lambda_2}^{\text{res}}$ carries the organization of the Financial sector. The eigenvalues $\lambda_4 = \lambda_5^*$ exhibit clustering of the Energy and the Consumer Staples sector. Therefore, the deviating eigenvalues can – in general – be associated with the clustering of different sectors. We note, that – in contrast to the original data – we did not find hub-like interactions in the data where the market-mode has been removed. From a methodological viewpoint, the close correspondence between the \mathbf{C}_{λ_i} and the different sectors visible in the network-representation of the data confirms the validity of our analysis and its tools. Only the negative eigenvalue $\text{Re}(\lambda_3) \approx -1$ and its contribution \mathbf{C}_{λ_3} is associated with time-lagged anti-correlations between various sectors which are naturally not visible in the network-view.

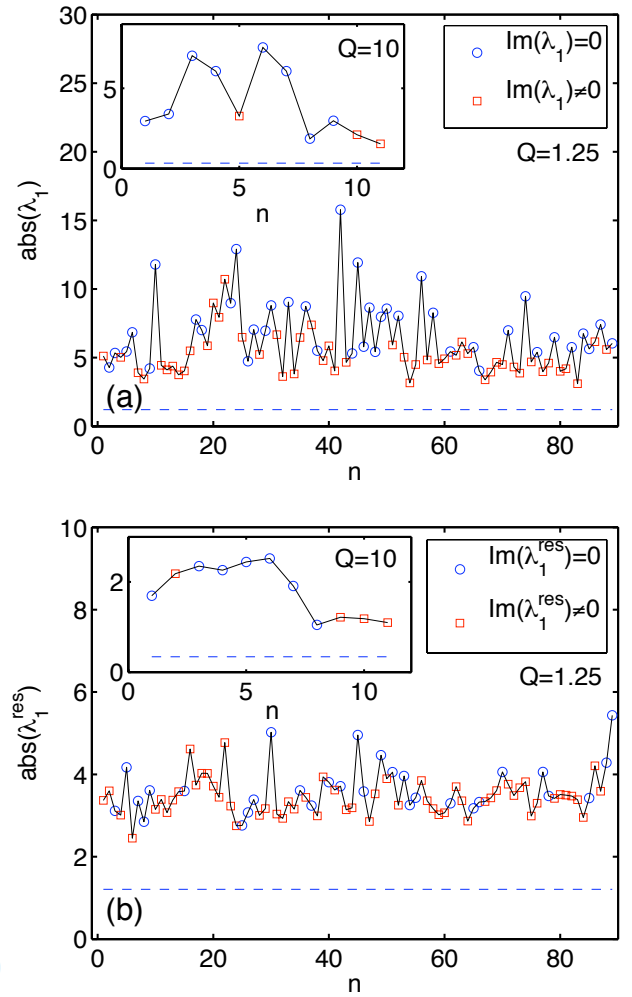


FIG. 9: (a) Time dependence of the largest eigenvalue of $\mathbf{C}_1(T_n)$ as a function of the period index n for $T = 500$ (main figure) and $T = 4000$ (inset). Values are plotted as blue circles if the largest eigenvalue is located on the real axis ($\text{Im}(\lambda_n^{\text{max}}) = 0$) and as red squares otherwise. (b) Same for $\mathbf{C}_{\tau}^{\text{res}}$.

4. TIME DEPENDENCE

After having discussed the interpretation of deviating eigenvalues, we now discuss the time-dependence of the correlation matrices as the last point of our analysis. Within the developed framework, we can immediately use the prediction of the support of the eigenvalue spectra in the complex plane \mathcal{C} to determine a minimum sampling period T (or equivalently a minimum value of Q) at which the estimated cross-correlations still exhibit non-random structure. This is possible since we know that if eigenvalues are outside the support the data is non-random. Reducing T too much one expects to arrive at a very noisy estimate of the lagged correlation matrix, which will manifest itself in having no departing eigen-

values at all.

We calculate $\mathbf{C}_1(T_i)$ for consecutive, non-overlapping time periods T_i and find that – very remarkably – down to a information to noise ratio of $Q \approx 1.25$, clear deviations from the predicted support occur. This means that even though noise is drastically increased for low values of Q , non-random structures prevail even at short time-scales.

More specifically, we analyzed 11 correlation matrices obtained from time slices of 4000 observations ($Q=10$), and 89 matrices for 500 time points each. For each individual sub-period T_n , we compute lagged correlation matrices $\mathbf{C}(T_n)$ for the raw data as well as on the matrices resulting from the regression model, $\mathbf{C}^{\text{res}}(T_n)$. Fig. 9 (a) shows a plot of the absolute value, $\text{abs}(\lambda_n)$, of the maximal eigenvalue found for each sub-period, indexed by n . The dashed blue line corresponds to the prediction of the support r_{max} . We immediately recognize that for $Q = 10$, as well as for $Q = 1.25$ the largest eigenvalue lies significantly above the noise regime. On the other hand, the absolute value of the largest eigenvalue is quite volatile and anti-persistent for $Q = 1.25$. We also observe that the largest eigenvalues with non-zero imaginary parts (red squares) mainly occur at low values of $\text{abs}(\lambda_n)$, whereas real eigenvalues occur at absolute values. If the eigenvalue is real, the lead-lag network is dominated by strong, approximately symmetric effects; for imaginary eigenvalues the network is dominated by asymmetric correlations, i.e. anti-correlations may play a distinctive part too. We find that if an eigenvalue λ_1 was real (i.e. marked by a blue circle in Fig. 9), the analysis of the preceding sections always identified the IT sector mainly contributing to \bar{u}_1 (for $Q = 10$). On the other hand, if the largest eigenvalue was imaginary, no unique interpretation appeared to be valid for all of the sub-periods. In Fig. 9b we show the same for our continuing antagonist \mathbf{X}^{res} . Again, we observe $\text{abs}(\lambda_n)$ being clearly located above the random frontier for all sub-periods. The movement of $\text{abs}(\lambda_1)$ is less volatile. Closer investigation of the underlying eigenvalues for $Q = 10$ revealed changing participation of the sectors (measured by the quantity I_{si} as defined in Eq. (22)). In effect, for all of the 11 sub-periods either the Energy (in periods 6-9) or the Utilities sector (in periods 3, 5) appeared as primarily contributing. In the rest of the periods, both of these sectors were represented strongly in I_{si} .

The last question addressed in this analysis is about the correlations of the lagged correlation matrices: Are significant lagged correlations only found *a posteriori* or does the data indicate a possibility for a reasonable prediction of future lead-lag structures? To this end we calculate the correlation of matrix elements between the lagged correlation matrices obtained from different (non-overlapping) observation periods T_n and T_m ,

$$c(T_n, T_m) = \frac{\langle (C_{\tau}^{ij}(T_n) - \langle C_{\tau}^{ij}(T_n) \rangle_{ij})(C_{\tau}^{ij}(T_m) - \langle C_{\tau}^{ij}(T_m) \rangle_{ij}) \rangle_{ij}}{\sigma_{T_n} \sigma_{T_m}}. \quad (24)$$

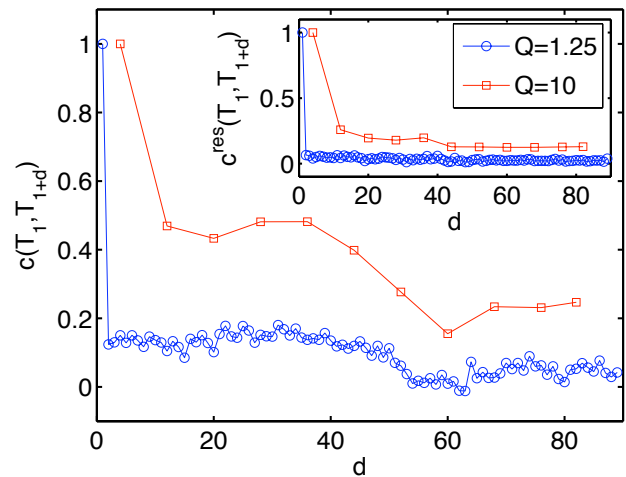


FIG. 10: Matrix element correlation $c(T_1, T_{1+d})$ as described in Eq. (24) for various time lags d for the original data and for the residuals \mathbf{X}^{res} .

Here, the average extends over all matrix-elements and σ_{T_n} denotes the standard deviation of matrix $\mathbf{C}_1(T_n)$. Fig. 10 depicts the characteristics we obtained from empirical data. While the expected band of correlation-coefficients would be bound by very small values (in the order of $1/400$), we find extremely significant correlations, especially for the $Q = 10$ case. As expected, the 'predictability' of future weighted lead-lag matrices is significantly higher for lagged matrices calculated over longer sub-periods. The inset of Fig. 10 shows $c^{\text{res}}(T_n, T_m)$, i.e. the same quantity calculated for the residual data. Overall correlations are lower in this case, meaning nothing else than that the market-wide movements exhibit predictable lead-lag structures. In both cases, one could make practical use of our results by performing a cleaning of the matrices (in analogy to the method described in [7]) and test whether or not the cleaned matrix at time T_n allows for an improved estimation of the future lagged correlation matrices at times $T_m > T_n$. We found that this is indeed the case (not shown) but a comprehensive analysis of this issue remains beyond the present scope of the paper.

5. CONCLUSION

We extended random matrix theory to lagged cross-correlation matrices and theoretically derived the eigenvalue spectra of the respective real, asymmetric random matrices as a function of the information to noise ratio, Q . Specifically, we have shown that an inverse Abel-transform can be used to reconstruct the radial density, $\rho(r)$, from re-scaled projections available from solutions of the symmetrized problem. Based on this previously unknown technique we were able to obtain theoretical

results of eigenvalue spectra, which we compared to empirical cross-correlations of 5 min returns of the S&P500. We explained the observed shift of the support and found that the distribution of eigenvalues in the bulk of the distribution can be considered as reasonably random. Based on these findings, we have shown that the eigenvalues deviating from the support carry information. We discussed various structural properties of these deviating eigenvalues. Analyzing data based on the residuals of a regression to the market-mode, we found that clustering in the lead-lag network is strongly enhanced. Looking at lagged correlation matrices pertaining to sub-periods of the overall investigation period we found that deviations from the theoretical prediction do occur at quite low information to noise ratios. We also found that significant parts of the lagged correlation matrix should be predictable via measurements of past (non-overlapping) periods. Based on the results of the present paper, a cleaning of the lagged correlation matrix can be performed which could directly lead to a number of practical applications.

We think that the current work can be extended in various directions. On the theoretical side, a closer investigation of the nature finite-size effects in the ensemble of time-lagged correlation matrices and comparison with the exact finite-size result of the random real asymmetric case [11] would be tempting. Regarding the mathematical difficulties of calculating eigenvalue spectra of asymmetric matrices, the proposed method of an inverse Abel-transform may prove helpful in many additional cases. We also think that some work is needed in an exact understanding of the relation between the eigenvalue spectra (including the left and right eigenvectors of the ensemble discussed here) and the singular value decomposition of related problems [15]. Also a rigorous study of the efficiency of different methods of cleaning could be pursued as well.

Finally we believe that the presented work – in general – should allow for an eigenvalue-dependent, systematic study of the influence of matrices and their interplay with equal time-correlations between financial assets in concrete models. The fact that cluster structure conforming with market sectors can be found in lagged correlation matrices already indicates the direction of findings to be expected from such work.

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Appendix A

Based on the series expansion (10) of the potential ϕ , we have calculated the first four terms in the series. For the first term, one easily obtains

$$\lim_{N \rightarrow \infty} \frac{1}{N} \langle \text{Tr}(\mathbf{B}) \rangle_c = \lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr}(\langle \mathbf{C}\mathbf{C}^T \rangle_c) = \frac{1}{Q} \quad (25)$$

since all other terms vanish as $\text{Tr}(\mathbf{C})$ gives just N times the averages of the autocorrelation of the assumed iid white noise process. In Eq. (25), we have also made use of the fact that the diagonal of $\mathbf{C}\mathbf{C}^T$ contains N -times the variances of correlations between white-noise processes ($\approx 1/T$). For calculating the second term, it is useful to remember $\text{Tr}(\mathbf{A}\mathbf{B}) = \text{Tr}(\mathbf{B}\mathbf{A})$ and $\text{Tr}(\mathbf{C}\mathbf{C}) = \text{Tr}(\mathbf{C}^T \mathbf{C}^T)$ as well as taking into account that odd powers of C vanish. One then arrives at

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \langle \text{Tr}(\mathbf{B}^2) \rangle_c &= \lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr}(\langle (\mathbf{C}\mathbf{C}^T)^2 \rangle_c) \\ &+ 2(x^2 + y^2) \text{Tr}(\langle \mathbf{C}\mathbf{C}^T \rangle_c) \\ &+ 2(x^2 - y^2) \text{Tr}(\langle \mathbf{C}\mathbf{C} \rangle_c) \quad (26) \end{aligned}$$

This structure is also typical for higher order terms (not shown for brevity). The trace in the 'dangerous' term proportional to $x^2 - y^2$ is nothing else than N times the variance of autocorrelations which is just $1/T$ for a Gaussian process. Thus, in total, the term vanishes as $1/T$ in the limit $N \rightarrow \infty$ with $Q = \text{const.}$, and one gets

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr}(\mathbf{B}^2)_c = K + 2rQ^{-1} \quad (27)$$

where $K = \lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr}(\langle (\mathbf{C}\mathbf{C}^T)^2 \rangle_c)$ is a constant (for $T > N$ one has $K \approx 2/Q^2$). In very similar calculations, it is easy (but tedious), to check that

$$\frac{1}{N} \langle \text{Tr}(\mathbf{B}^3) \rangle = f(r) \quad \text{and} \quad \frac{1}{N} \langle \text{Tr}(\mathbf{B}^4) \rangle = g(r) \quad (28)$$

The typical situation for higher order terms is similar to the one for the second order term, i.e. the terms in r generally depend on some function of Q and the 'dangerous' terms (like $(x^2 - y^2)^2$) vanish since they remain constant for growing matrix size and are thus neutralized by the prefactor $1/N$. We do not expect any different behavior for terms higher than fourth order.

Appendix B

The uniform eigenvalue distribution of real asymmetric matrices in the complex plane \mathcal{C} found in [30] can be almost trivially recovered from Wigner's semicircle law of real symmetric matrices via application of the inverse Abel-transform. Starting from Wigner's semicircle law $\rho(\bar{\lambda}) = \frac{1}{2\pi} \sqrt{4 - \bar{\lambda}^2}$ and after proper rescaling and normalization we may insert $\rho^S(\sqrt{2}x) = \frac{1}{\pi} \sqrt{2 - x^2}$ into Eq.

(15) to arrive at

$$\begin{aligned}\rho(r) &= \frac{1}{\pi^2} \int_r^{\sqrt{2}} \frac{x}{\sqrt{2-x^2}\sqrt{x^2-r^2}} dx \\ &= -\frac{1}{\pi^2} \arctan \left(\frac{\sqrt{2-x^2}}{\sqrt{x^2-r^2}} \right) \Big|_r^{\sqrt{2}} = \frac{1}{2\pi} .\end{aligned}\quad (29)$$

We immediately arrive at the result of an uniform eigenvalue distribution,

$$\rho(r) = \begin{cases} \frac{1}{2\pi} & 0 < r < \sqrt{2} \\ 0 & \text{elsewhere} \end{cases} . \quad (30)$$

Appendix C

For $Q = 1$, one solution can be written in the form

$$G_{r=1}^H(z) = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{\sqrt{z^2-4}}{z}} . \quad (31)$$

Note, that this equation shows a simple relation to the resolvent of the Gaussian orthogonal ensemble ($G_{Q=1}^S = \sqrt{\frac{1}{z}} G_{r=1}^{GOE}(z)$). The eigenvalue spectrum following from Eqs. (13) and (31) can then be written as

$$\begin{aligned}\rho_{Q=1}(\bar{\lambda}) &= \frac{1}{\sqrt{2}\pi} \frac{\sqrt{-\frac{1}{2} + \frac{2}{2+|\bar{\lambda}|}}}{\sqrt{\frac{|\bar{\lambda}|}{2+|\bar{\lambda}|}}} \\ &= \frac{1}{\sqrt{2}\pi} \sqrt{-\frac{1}{2} - \frac{1}{|\bar{\lambda}|} + \frac{2}{2+|\bar{\lambda}|} + \frac{4}{|\bar{\lambda}|(2+|\bar{\lambda}|)}} ,\end{aligned}\quad (32)$$

and is valued on the support $[-2, 2]$. After proper rescaling and taking an expression of the inverse Abel Transform equivalent to Eq. (15), namely

$$\rho_{Q=1}(r) = -\frac{1}{\pi r} \frac{d}{dr} \int_r^\infty x \frac{\rho_{Q=1}^S(\sqrt{2}x)}{\sqrt{x^2-r^2}} dx , \quad (33)$$

we end up with the expression

$$\rho_{Q=1}(r) = -\frac{\sqrt{2}}{\pi^2 r} \frac{d}{dr} \int_r^{\sqrt{2}} x \frac{\sqrt{\frac{\sqrt{2}}{x} - 1}}{\sqrt{x^2-r^2}} dx , \quad (34)$$

which can be evaluated to

$$\begin{aligned}\rho_{Q=1}(r) &= \frac{1}{K} \left[2^{3/4} 3r \Gamma\left(\frac{5}{4}\right) \Gamma\left(\frac{5}{4}\right) \Phi_2^1\left(\frac{1}{4}, \frac{5}{4}, \frac{3}{2}, \frac{\lambda^2}{2}\right) \right. \\ &\quad \left. - 2^{1/4} \Gamma\left(-\frac{1}{4}\right) \Gamma\left(\frac{7}{4}\right) \Phi_2^1\left(-\frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{\lambda^2}{2}\right) \right] ,\end{aligned}\quad (35)$$

where $K = 6\sqrt{\pi^5 r^3}$, $\Gamma(x)$ denotes the Gamma-Function and $\Phi_2^1(a, b, c, x)$ is the hypergeometric function. It can be checked, that – of course – $\int_0^{\sqrt{2}} 2\pi r \rho(r) dr = 1$.

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- [41] See e.g. [35] for a discussion of disc-annulus phase transition in the case of *non-hermitian* matrix models. In the present case the transition can be intuitively understood in the log-gas analogy as redundant factors 'push away' the other eigenvalues from the center.
- [42] In the case of the GinUE, where the accumulation of eigenvalues on the real axis is not present, our method may prove even more useful.
- [43] The effect of varying the time-difference aspect of lagged correlations has been carefully studied in [27], and we shall not discuss this issue here.